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On an Unboundedness Conjecture for Strong Unicity Constants*

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1. INTRODUCTION

Let $C(I)$ be the set of continuous, real-valued functions on the interval $I = [a, b]$, and let Π_n be the set of real polynomials of degree n or less. Let $\|\cdot\|$ denote the uniform norm on $C(I)$. For $f \in C(I)$ with best uniform approximation $T_n(f)$ from Π_n there is a positive constant r such that

$$\| p - T_n(f)\| \leqslant r(\| f - p \| - \| f - T_n(f)\|) \tag{1}$$

for all $p \in \Pi_n$. Inequality (1) is the strong unicity theorem due to Newman and Shapiro [6]. The strong unicity constant $M_n(f)$ is defined to be the smallest positive constant r such that (1) is true for all $p \in \Pi_n$.

The dependence of $M_n(f)$ on f, n, and I has been the subject of several recent papers (see [3, 4, 7] and the references of [4]). This paper concerns the dependence of $M_n(f)$ on n. The problem of characterizing those $f \in C(I)$ for which the sequence

$$\{M_n(f)\}_{n=0}^{\infty} \tag{2}$$

is bounded was posed by Poreda [7]. It is easy to see that if $f \in \Pi_m$, then $M_n(f) = 1$ for all $n \geqslant m$ and, hence, (2) is bounded. In his paper, Poreda constructs a function $f \in C(I)$ for which the sequence (2) is unbounded. Henry and Roulier [4] demonstrate a class of functions quite different from Poreda's example for which

$$\lim_{n\to\infty} M_n(f) = \infty$$

and conjecture that the sequence (2) remains bounded only if f is a polynomial.

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The results of this paper extend both Poreda's example and the result of Henry and Roulier to form a wider class of functions whose sequence of strong unicity constants is known to be unbounded. In particular, Poreda's problem is reduced to considering only those nonpolynomial functions f for which the extremal set of $f - T_n(f)$ eventually contain more than $n + 2$ points.

In view of the results of Henry and Roulier, it may be interesting to determine if a nonpolynomial $f \in C(I)$ exists for which (2) has a bounded subsequence. We answer this question by constructing an $f \in C(I)$ for which $\underline{\lim}_{n\to\infty} M_n(f) = 1$ and $\overline{\lim}_{n\to\infty} M_n(f) = \infty$.

2. The Unboundedness of $M_n(f)$

Let $f \in C(I)$. For each n, let $S_n = \{p \in \Pi_n : \|p\| = 1\}$, $e_n(f) = f - T_n(f)$, and $E_n(f) = \{x \in I: |e_n(f)(x)| = \|e_n(f)\|\}$. We refer to $E_n(f)$ as the extremal set of $e_n(f)$. The following characterization of the strong unicity constant appears in the papers [1, 5]. If $f \notin \Pi_n$, then

$$M_n(f) = \{\min_{p\in S_n} \max_{x\in E_n(f)} [\operatorname{sgn} e_n(f)(x)]\, p(x)\}^{-1}. \tag{3}$$

In Poreda's example [7], there is an interval J properly contained in I such that $E_n(f) \subseteq J$ for infinitely many n. In Theorem 1, we show that whenever the $E_n(f)$ do not "fill out" the interval I (as in Poreda's example) the sequence (2) is unbounded.

THEOREM 1. *Let $f \in C(I)$ and suppose there is a nondegenerate interval $[c, d] \subseteq I$ and a strictly increasing sequence $\{n_\alpha\}_{\alpha=1}^{\infty}$ of positive integers such that $E_{n_\alpha}(f) \cap [c, d] = \varnothing$ for $a = 1, 2,\ldots$. Then $\{M_n(f)\}_{n=0}^{\infty}$ is unbounded.*

Proof. We may assume that $a < c < d < b$. Let $e = (c + d)/2$. Let $0 < \delta < 1$ be arbitrary. By a theorem due to Wolibner (see [7, 9]), there is a polynomial q such that $q(a) = 0$, $q(c) = \delta$, $q(e) = 1$, $q(d) = \delta$, $q(b) = 0$, and q is monotone on each of the intervals $[a, c]$, $[c, e]$, $[e, d]$, and $[d, b]$. Thus $\|q\| = 1$ and $|q(x)| < \delta$ on $I\setminus[c, d]$. We now select an α such that n_α is greater than the degree of q. Thus $q \in S_{n_\alpha}$ and

$$\max_{x\in E_{n_\alpha}(f)} [\operatorname{sgn} e_{n_\alpha}(f)(x)]\, q(x) < \delta.$$

Hence,

$$0 < \min_{p\in S_{n_\alpha}} \max_{x\in E_{n_\alpha}(f)} [\operatorname{sgn} e_{n_\alpha}(f)(x)]\, p(x) \leqslant \delta$$

and by (3), $M_{n_\alpha}(f) \geqslant 1/\delta$. Thus the sequence (2) is unbounded, and Theorem 1 is proven.

The analysis of Henry and Roulier [4] utilizes a characterization of $M_n(f)$ different from (3). If $f \in C(I)$, the alternation theorem [2, p. 75] asserts that there are $n + 2$ points

$$x_{0n} < x_{1n} < \cdots < x_{(n+1)n} \tag{4}$$

in $E_n(f)$ on which the error function $e_n(f)$ alternates in sign. For $k = 0,\ldots, n+1$, let q_{kn} be the polynomial in Π_n such that $q_{kn}(x_{in}) = (-1)^i$, $i = 0,\ldots, n+1$, $i \neq k$. Cline [3] has shown that

$$K_n = K_n(x_{0n}, \ldots, x_{(n+1)n}) = \max_{0 \leqslant k \leqslant n+1} \| q_{kn} \| \tag{5}$$

is a suitable strong unicity constant, that is, $K_n \geqslant M_n(f)$. Henry and Roulier [4] remark that if $E_n(f)$ contains exactly $n + 2$ points, then $K_n = M_n(f)$. Using this characterization of the strong unicity constant, Henry and Roulier prove the following theorem.

THEOREM 2. *Let $f \in C^\infty(I)$. Suppose there exist an $\epsilon > 0$ and an $N > 0$ such that for all $n \geqslant N$, $f^{(n+1)}$ is positive on (a, b) and*

$$\frac{f^{(n+1)}(\xi)}{f^{(n+1)}(\eta)} \geqslant \epsilon$$

for all $\xi, \eta \in I$. Then $\lim_{n\to\infty} M_n(f) = \infty$.

In their proof of Theorem 2, Henry and Roulier require the alternation set (4) to be distributed throughout the interval I in a particular fashion to show that $\lim_{n\to\infty} K_n = \infty$. We prove that $\lim_{n\to\infty} K_n = \infty$ regardless of how (4) is distributed throughout I.

THEOREM 3. *Suppose that for each n, there are $n + 2$ points $x_{0n} < x_{1n} < \cdots < x_{(n+1)n}$ in I given. Then $\lim_{n\to\infty} K_n = \infty$, where $K_n = K_n(x_{0n}, \ldots, x_{(n+1)n})$ is given by (5).*

Proof. For the sake of notation, we show that the sequence $\{K_n\}_{n=0}^\infty$ is unbounded and note that the following analysis can be used to show that every subsequence of $\{K_n\}_{n=0}^\infty$ is unbounded. For convenience let $x_{-1n} = a$ and $x_{(n+2)n} = b$.

We require the following lemma.

LEMMA. *There is a strictly increasing sequence $\{n_\alpha\}_{\alpha=1}^\infty$ of positive integers such that for each α there is a $k_\alpha \in \{0,\ldots, n_\alpha + 1\}$ and there is a $P_\alpha \in \Pi_{n_\alpha}$ with $|p_\alpha(x_{in_\alpha})| \leqslant 1$, $i = 0,\ldots, n_\alpha + 1$, $i \neq k_\alpha$ where*

$$\lim_{\alpha\to\infty} \max_{x \in [x_{(k_\alpha - 1)n_\alpha}, x_{(k_\alpha+1)n_\alpha}]} |p_\alpha(x)| = \infty.$$

Proof. For each n, let L_n denote the interpolation operator which assigns to each $g \in C(I)$ the polynomial $L_n g$ in Π_n which fits g at the nodes $x_{1n}, \ldots, x_{(n+1)n}$. From Rivlin [8, Theorem 4.2, p. 91, proof of Theorem 4.3, p. 92], there is a sequence $\{f_n\}_{n+1}^{\infty}$ in $B = \{g \in C(I) : \| g \| \leqslant 1\}$ such that

$$\| L_n f_n \| \geqslant \frac{4}{\pi^2} \log(n) - 1$$

for all n. Thus $\lim_{n \to \infty} \| L_n f_n \| = \infty$.

If the numbers

$$\max_{x \in [x_{-1n}, x_{1n}]} |(L_n f_n)(x)| \tag{6}$$

are unbounded with respect to n, then there is a strictly increasing sequence $\{n_\alpha\}_{\alpha=1}^{\infty}$ of positive integers such that

$$\lim_{\alpha \to \infty} \max_{x \in [x_{-1n_\alpha}, x_{1n_\alpha}]} |(L_{n_\alpha} f_{n_\alpha})(x)| = \infty.$$

In this case, we let $k_\alpha = 0$ and $p_\alpha = L_{n_\alpha} f_{n_\alpha}$.

In case the numbers (6) are bounded with respect to n, there is a number $A \geqslant 1$ such that

$$\max_{x \in [x_{-1n}, x_{1n}]} |(L_n f_n)(x)| \leqslant A$$

for all n. Let $g_n = f_n / A$. Then $g_n \in B$, $|(L_n g_n)(x_{in})| \leqslant 1$, $i = 0, \ldots, n+1$, and $\lim_{n \to \infty} \| L_n g_n \| = \infty$. In this case, we let $\{n_\alpha\}_{\alpha=1}^{\infty}$ be the identity sequence and discard the subsequence notation. For each n, let $y_n \in I$ be such that $|(L_n g_n)(y_n)| = \| L_n g_n \|$ and select $k_n \in \{0, \ldots, n+1\}$, where $y_n \in [x_{(k_n - 1)n}, x_{(k_n + 1)n}]$. We now choose $p_n = L_n g_n$ and note that

$$\lim_{n \to \infty} \max_{x \in [x_{(k_n - 1)n}, x_{(k_n + 1)n}]} | p_n(x)| = \infty.$$

Thus the lemma is proved.

We now return to the proof of Theorem 3. Choose a sequence $\{n_\alpha\}_{\alpha=1}^{\infty}$ with corresponding $k_\alpha \in \{0, \ldots, n_\alpha + 1\}$ and $p_\alpha \in \Pi_{n_\alpha}$ as in the above lemma. For $i = 0, \ldots, n_\alpha + 1$, $i \neq k_\alpha$, let l_{in_α} be the polynomial in Π_{n_α} such that $l_{in_\alpha}(x_{in_\alpha}) = 1$ and $l_{in_\alpha}(x_{jn_\alpha}) = 0$, $j = 0, \ldots, n_\alpha + 1$, $j \neq i$, $j \neq k_\alpha$. It can be shown that $(-1)^i \, l_{in_\alpha}(x)$, $i = 0, \ldots, n_\alpha + 1$, $i \neq k_\alpha$, have the same sign on the interval $(x_{(k_\alpha - 1)n_\alpha}, x_{(k_\alpha + 1)n_\alpha})$. For all sufficiently large α, we may select $y_\alpha \in (x_{(k_\alpha - 1)n_\alpha}, x_{(k_\alpha + 1)n_\alpha})$ such that

$$| p_\alpha(y_\alpha)| = \max_{x \in [x_{(k_\alpha - 1)n_\alpha}, x_{(k_\alpha + 1)n_\alpha}]} | p_\alpha(x)|.$$

Thus

$$\begin{aligned}\max_{x\in[x_{(k_\alpha-1)n_\alpha},x_{(k_\alpha+1)n_\alpha}]} |p_\alpha(x)| &= |p_\alpha(y_\alpha)| \\ &= \left|\sum_{\substack{i=0\\ i\neq k_\alpha}}^{n_\alpha+1} p_\alpha(x_{in_\alpha})\, l_{in_\alpha}(y_\alpha)\right| \\ &\leq \sum_{\substack{i=0\\ i\neq k_\alpha}}^{n_\alpha+1} |l_{in_\alpha}(y_\alpha)| \\ &= \left|\sum_{\substack{i=0\\ i\neq k_\alpha}}^{n_\alpha+1} (-1)^i\, l_{in_\alpha}(y_\alpha)\right| \\ &= |q_{k_\alpha n_\alpha}(y_\alpha)| \\ &\leq \| q_{k_\alpha n_\alpha} \| \\ &\leq K_{n_\alpha}.\end{aligned}$$

Thus

$$\lim_{\alpha\to\infty} K_{n_\alpha} = \infty \tag{7}$$

and $\{K_n\}_{n=0}^{\infty}$ is unbounded.

A reflection of the lemma indicates that every strictly increasing sequence of positive integers has a subsequence $\{n_\alpha\}_{\alpha=1}^{\infty}$ which satisfies the properties of the Lemma and thus (7). As a result, every subsequence of $\{K_n\}_{n=0}^{\infty}$ is unbounded and, therefore, $\lim_{n\to\infty} K_n = \infty$. The proof of Theorem 3 is now complete.

In view of the remark of Henry and Roulier [4] that $M_n(f) = K_n$ whenever $E_n(f)$ contains exactly $n + 2$ points, the next theorem follows from Theorem 3.

THEOREM 4. *Let $f \in C(I)$. If $E_n(f)$ contains exactly $n + 2$ points for infinitely many n, then the sequence $\{M_n(f)\}_{n=0}^{\infty}$ is unbounded.*

Proof. Let $\{n_\alpha\}_{\alpha=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $E_{n_\alpha}(f)$ contains exactly $n_\alpha + 2$ points, say $x_{0n_\alpha} < x_{1n_\alpha} < \cdots < x_{(n_\alpha+1)n_\alpha}$. By the remark of Henry and Roulier [4], $M_{n_\alpha}(f) = K_{n_\alpha}(x_{0n_\alpha},\ldots, x_{(n_\alpha+1)n_\alpha})$ and by Theorem 3 $\lim_{\alpha\to\infty} M_{n_\alpha}(f) = \infty$. Thus Theorem 4 is proved.

Theorem 4 reduces the problem in [7] to considering only those $f \in C(I)$ such that for all sufficiently large n the set $E_n(f)$ contains more than $n + 2$ points. It should be remarked that it is unknown, at least to the author, whether such functions exist. A condition which ensures that $E_n(f)$ contains exactly $n + 2$ points is that $f^{(n+1)}$ does not vanish on the open interval (a, b).

This observation and Theorem 3 allow us to remove the rather stringent condition on $f^{(n+1)}$ in Theorem 2.

THEOREM 5. *Let $f \in C[a, b] \cap C^{\infty}(a, b)$. If there is an $N > 0$ such that for all $n \geqslant N$, $f^{(n+1)}$ does not vanish in the open interval (a, b), then* $\lim_{n\to\infty} M_n(f) = \infty$.

In concluding this section, we remark that the conditions of Theorem 5 are satisfied by a variety of functions which do not satisfy the conditions of Theorem 2, for example, $\sin(x)$ and $\cos(x)$ with $I = [0, \pi/2]$, $\exp(x^2)$ with $I = [0, 1]$, and $\log(x)$ with $I = [a, b]$, where $0 < a < b$.

3. AN EXAMPLE

In this section, we demonstrate a function $f \in C(I)$ for which $\underline{\lim}_{n\to\infty} M_n(f) = 1$ (which is minimal) and $\overline{\lim}_{n\to\infty} M_n(f) = \infty$. This example is based on a construction of Poreda [7] in which he shows that given an increasing sequence $\{x_\alpha\}_{\alpha=0}^{\infty}$ of points in I, there is an $f \in C(I)$ whose extremal sets $E_n(f)$ are contained in $\{x_\alpha\}_{\alpha=0}^{\infty}$ for infinitely many n. In the following example, the extremal sets deviate considerably from those of Poreda's construction.

THEOREM 6. *There is a function $f \in C(I)$ such that $\underline{\lim}_{n\to\infty} M_n(f) = 1$ and $\overline{\lim}_{n\to\infty} M_n(f) = \infty$.*

Proof. Since Π_n is finite dimensional and $S_n = \{p \in \Pi_n : \| p \| = 1\}$ is bounded, S_n is equicontinuous. Thus for each n, there is a $\delta_n > 0$ such that $| p(x) - p(y)| < 1/n$ for all $p \in S_n$ and all $x, y \in I$ where $| x - y | < \delta_n$.

We construct sequences $\{Q_\alpha\}_{\alpha=1}^{\infty}$, $\{n_\alpha\}_{\alpha=0}^{\infty}$, $\{m_\alpha\}_{\alpha=0}^{\infty}$, and $\{X_\alpha\}_{\alpha=0}^{\infty}$ recursively as follows. Let $n_0 = 1$, and select $m_0 > n_0$ and a point set

$$X_0 : \quad a = x_{00} < x_{10} < \cdots < x_{(m_0+1)0} = b$$

such that $\max_{1\leqslant i\leqslant m_0+1} (x_{i0} - x_{(i-1)0}) < \delta_{n_0}$. In the induction stage, suppose that Q_α (if $\alpha \geqslant 1$), n_α, m_α, and

$$X_\alpha : \quad a = x_{0\alpha} < x_{1\alpha} < \cdots < x_{(m_\alpha+1)\alpha} = b$$

have been found. We select a polynomial $Q_{\alpha+1}$ via Wolibner's theorem [7, 9] such that $Q_{\alpha+1}(x_{i\alpha}) = (-1)^i\, 2^{-(\alpha+1)}$, $i = 0,\ldots, m_\alpha + 1$, and $Q_{\alpha+1}$ is monotone on each of the intervals $[x_{(i-1)\alpha}, x_{i\alpha}]$, $i = 1,\ldots, m_\alpha + 1$. Let $n_{\alpha+1}$ be the degree of $Q_{\alpha+1}$. Since $Q_{\alpha+1}$ has at least $m_\alpha + 1$ zeros, $n_{\alpha+1} > m_\alpha$. Select $m_{\alpha+1} > n_{\alpha+1}$ and a point set

$$X_{\alpha+1}: \quad a = x_{0(\alpha+1)} < x_{1(\alpha+1)} < \cdots < x_{(m_{\alpha+1}+1)(\alpha+1)} = b$$

such that (i) $X_\alpha \subseteq X_{\alpha+1}$, (ii) for $i = 1,..., m_\alpha + 1$, $\{x \in X_{\alpha+1} : x_{(i-1)\alpha} < x < x_{i\alpha}\}$ contains an even number of points, and (iii) $\max_{1 \leqslant i \leqslant m_{\alpha+1}+1} (x_{i(\alpha+1)} - x_{(i-1)(\alpha+1)}) < \delta_{n_{\alpha+1}}$.

Let $f = \sum_{\alpha=1}^{\infty} Q_\alpha$. Since each $\| Q_\alpha \| = 2^{-\alpha}$, the Weierstrass M-test insures that $f \in C(I)$. The strict monotonicity of each Q_α on the subintervals of I induced by $X_{\alpha-1}$ and properties (i) and (ii) imply that for $\beta \geqslant 1$, $| \sum_{\alpha=\beta+1}^{\infty} Q_\alpha |$ has norm $2^{-\beta}$ and attains its norm only on X_β. Furthermore, $\sum_{\alpha=\beta+1}^{\infty} Q_\alpha$ alternates in sign at the points of X_β. By the alternation theorem [2, p. 75] and since $n_\alpha < m_\alpha < n_{\alpha+1}$ for all α, $T_{n_\beta}(f) = T_{m_\beta}(f) = \sum_{\alpha=1}^{\beta} Q_\alpha$ and $E_{n_\beta}(f) = E_{m_\beta}(f) = X_\beta$ for all $\beta \geqslant 1$.

For any $p \in S_{n_\beta}$, select $y \in I$ such that $|p(y)| = 1$. By (iii) there is an $x_{i\beta} \in E_{n_\beta}(f) = X_\beta$ such that sgn $e_{n_\beta}(f)(x_{i\beta}) =$ sgn $p(y)$ and $| y - x_{i\beta} | < \delta_{n_\beta}$. Thus

$$\begin{aligned}\max_{x \in E_{n_\beta}(f)} [\operatorname{sgn} e_{n_\beta}(f)(x)]\, p(x) &\geqslant [\operatorname{sgn} e_{n_\beta}(f)(x_{i\beta})]\, p(x_{i\beta}) \\ &= [\operatorname{sgn} p(y)]\, p(y) - [\operatorname{sgn} p(y)](p(y) - p(x_{i\beta}) \\ &> 1 - 1/n_\beta .\end{aligned}$$

By [2, problem 6, p. 83] and by (3)

$$1 \leqslant M_{n_\beta}(f) \leqslant (1 - 1/n_\beta)^{-1}$$

for all $\beta \geqslant 1$. Hence, $\underline{\lim}_{n\to\infty} M_n(f) = 1$. Now note that $E_{m_\beta}(f) = X_\beta$ contains exactly $m_\beta + 2$ points for all $\beta \geqslant 1$. By Theorem 4, the sequence (2) corresponding to this function f is unbounded, and, as a result, $\overline{\lim}_{n\to\infty} M_n(f) = \infty$.

4. Conclusions

The results of this paper tend to strengthen the conjecture of Henry and Roulier [4] that the sequence (2) is unbounded for all nonpolynomial $f \in C(I)$ and reduce the problem of Poreda [7] to a degenerate case where the extremal sets contain "too many" points. An interesting problem which arises from this analysis is that of determining whether a function $f \in C(I)$ exists such that for all sufficiently large n, $E_n(f)$ contains more than $n + 2$ points. We note that answering this question in the negative would completely solve Poreda's problem.

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